

Asymptotic distribution of a consistent cross-spectrum
estimator based on uniformly spaced samples of a
non-bandlimited process

Radhendushka Srivastava and Debasis Sengupta

Applied Statistics Unit
Indian Statistical Institute
203 B.T. Road
Kolkata 700108, India

(e-mail: radhe_r@isical.ac.in; sdebasis@isical.ac.in).

June 9, 2010

Asymptotic distribution of a consistent cross-spectrum estimator based on uniformly spaced samples of a non-bandlimited process

Abstract

It is well known that if the power spectral density of a continuous time stationary stochastic process does not have a compact support, data sampled from that process at any uniform sampling rate leads to biased and inconsistent spectrum estimators. In a recent paper, the authors showed that the smoothed periodogram estimator can be consistent, if the sampling interval is allowed to shrink to zero at a suitable rate as the sample size goes to infinity. In this paper, this ‘shrinking asymptotics’ approach is used to obtain the limiting distribution of the smoothed periodogram estimator of spectra and cross-spectra. It is shown that, under suitable conditions, the scaling that ensures weak convergence of the estimator to a limiting normal random vector can range from cube-root of the sample size to square-root of the sample size, depending on the strength of the assumption made. The results are used to construct asymptotic confidence intervals for spectra and cross spectra. It is shown through a Monte-Carlo simulation study that these intervals have appropriate empirical coverage probabilities at moderate sample sizes.

Keywords: Power spectral density, spectrum estimation, smoothed periodogram, shrinking asymptotics, asymptotic confidence interval.

1 Introduction

Estimation of power spectral density (spectrum) of a continuous time, mean square continuous, stationary stochastic process is a classical problem. Generally the estimation is based on finitely many samples of the process. It is well known that if the spectrum is compactly supported (bandlimited), then it can be estimated from uniformly spaced samples, provided the sampling is done at the Nyquist rate or faster (Kay, 1999). For sampled non-bandlimited processes, or bandlimited processes sampled at sub-Nyquist rate, the problem of aliasing leads to biased estimation. For this reason, it is sometimes argued that a non-bandlimited spectrum can never be estimated consistently from uniformly spaced samples at any fixed sampling rate (Shapiro and Silverman, 1960; Masry, 1978).

Consequently, some researchers have turned to non-uniform sampling schemes such as stochastic sampling and periodic non-uniform sampling. Masry (1978) proved the consistency of some

spectrum estimators based on stochastic sampling schemes, under appropriate conditions that allow the underlying spectrum to be non-bandlimited. For bandlimited processes, it has been shown that periodic non-uniform sampling at sub-Nyquist average rate can lead to consistent spectrum estimation (Marvasti, 2001).

It is important to note that the argument of inconsistency of spectrum estimators computed from uniformly spaced samples is based on the assumption that the sampling rate remains fixed even as the sample size goes to infinity. However, when one has the resources to increase the sample size indefinitely, one would like to use some of those resources to sample faster, rather than being constrained by a fixed sampling rate. In fact, it has been shown (Srivastava and Sengupta, 2010) that if this constraint is removed, and the sampling rate is allowed to increase suitably as the sample size goes to infinity, then the smoothed periodogram can be a consistent estimator of a non-bandlimited spectral density.

It should be noted that uniform sampling is generally far easier to implement than non-uniform sampling. For this reason, the fact of consistency of a spectrum estimator computed from uniformly spaced samples of a non-bandlimited process is noteworthy. This fact also gives rise to further questions about the estimator, such as its convergence in distribution, and construction of asymptotic confidence intervals for the spectrum, based on the estimator. These questions have so far not been addressed through asymptotic calculations that allow the sampling rate to go to infinity. This is what we propose to do in this paper.

The asymptotic approach chosen here (referred to as ‘shrinking asymptotics’ by Fuentes, 2002) was also adopted by other authors (e.g., Constantine and Hall, 1994; Hall et al., 1994; Lahiri, 1999), although the asymptotic distribution of the smoothed periodogram has not been studied previously. This approach is different from the ‘fixed-domain asymptotics’ or ‘infill asymptotics’ approach (Chen et al., 2000; Stein, 1995; Zhang and Zimmerman, 2005; Lim and Stein, 2008) which, in the present case, would have required that the time-span of the original continuous-time data (before sampling) remains fixed as the sampling rate goes to infinity.

Let $\mathbf{X} = \{\mathbf{X}(t), -\infty < t < \infty\}$ be a vector-valued mean square continuous stationary stochastic process, having zero mean. We denote the components of the process \mathbf{X} by $X_a = \{X_a(t), -\infty < t < \infty\}$ for $a \in \{1, 2, \dots, r\}$, and the variance-covariance matrix of the process \mathbf{X} at lag τ by

$$\mathbf{C}(\tau) = \begin{pmatrix} C_{11}(\tau) & C_{12}(\tau) & \dots & C_{1r}(\tau) \\ C_{21}(\tau) & C_{22}(\tau) & \dots & C_{2r}(\tau) \\ \vdots & \vdots & & \vdots \\ C_{r1}(\tau) & C_{r2}(\tau) & \dots & C_{rr}(\tau) \end{pmatrix},$$

where

$$C_{a_1 a_2}(\tau) = E[X_{a_1}(t + \tau)X_{a_2}(t)] \text{ for } a_1, a_2 \in \{1, 2, \dots, r\}.$$

The spectral and cross-spectral density matrix of the process \mathbf{X} is denoted by

$$\Phi(\cdot) = \begin{pmatrix} \phi_{11}(\cdot) & \phi_{12}(\cdot) & \dots & \phi_{1r}(\cdot) \\ \phi_{21}(\cdot) & \phi_{22}(\cdot) & \dots & \phi_{2r}(\cdot) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{r1}(\cdot) & \phi_{r2}(\cdot) & \dots & \phi_{rr}(\cdot) \end{pmatrix}$$

where

$$\phi_{a_1 a_2}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{a_1 a_2}(t) e^{-it\lambda} dt \text{ for } -\infty < \lambda < \infty, \quad a_1, a_2 \in \{1, 2, \dots, r\}.$$

In this paper, we consider the following estimator of $\phi_{a_1 a_2}(\lambda)$ for $a_1, a_2 \in \{1, 2, \dots, r\}$:

$$\hat{\phi}_{a_1 a_2}(\lambda) = \frac{1}{2\pi n \rho_n} \sum_{t_1=1}^n \sum_{t_2=1}^n K(b_n(t_1 - t_2)) X_{a_1}\left(\frac{t_1}{\rho_n}\right) X_{a_2}\left(\frac{t_2}{\rho_n}\right) e^{-\frac{i(t_1 - t_2)\lambda}{\rho_n}} 1_{[-\pi\rho_n, \pi\rho_n]}(\lambda), \quad (1)$$

where $K(\cdot)$ is a covariance averaging kernel, b_n is the kernel bandwidth, ρ_n is the sampling rate and $1_A(\lambda)$ is the indicator of the event $\lambda \in A$.

In Section 2, we establish the consistency of the spectrum estimator (1) for non-bandlimited processes, which is a generalization of a result of Srivastava and Sengupta (2010) to the case of multivariate time series. It paves the way for our main result on the asymptotic distribution of the estimator, given in Section 3. Section 4 contains some discussion on optimal rates of convergence. In Section 5, we investigate the question as to how large the sample size should be in order for the applicability of the asymptotic distribution and the resulting pointwise confidence intervals. We look for answers through a Monte Carlo simulation study and report the findings. All the proofs are given in the appendix.

2 Consistency

In order to establish the consistency of the estimator $\hat{\phi}_{a_1 a_2}(\cdot)$ given in (1), we make a few assumptions on the process \mathbf{X} , the kernel $K(\cdot)$ and the sequences b_n and ρ_n .

ASSUMPTION 1. The function $g_{a_1 a_2}(\cdot)$, defined over the real line as $g_{a_1 a_2}(t) = \sup_{|s| \geq |t|} |C_{a_1 a_2}(s)|$ is integrable for all $a_1, a_2 \in \{1, 2, \dots, r\}$.

ASSUMPTION 2. The covariance averaging kernel function $K(\cdot)$ is continuous, even, square integrable and bounded by a non-negative, even and integrable function having a unique maximum at 0. Further, $K(0) = 1$.

ASSUMPTION 3. The kernel window width is such that $nb_n \rightarrow \infty$ as $n \rightarrow \infty$.

ASSUMPTION 4. The sampling rate is such that $\rho_n \rightarrow \infty$ and $\rho_n b_n \rightarrow 0$ as $n \rightarrow \infty$.

Note that Assumption 4 implies that $b_n \rightarrow 0$ as $n \rightarrow \infty$.

THEOREM 1. *Under Assumptions 1–4, the bias of the estimator $\hat{\phi}_{a_1 a_2}(\cdot)$ tends to zero uniformly over any closed and finite interval.*

In order to establish convergence of the variance-covariance matrix, we need a further assumption involving cumulants. Recall that the r -th order joint cumulant of the random variable (Y_1, \dots, Y_r) is given by

$$\text{cum}(Y_1, \dots, Y_r) = \sum_{\nu} (-1)^{p-1} (p-1)! \left(E \prod_{j \in \nu_1} Y_j \right) \times \dots \times \left(E \prod_{j \in \nu_p} Y_j \right), \quad (2)$$

where the summation is over all partitions $\nu = (\nu_1, \dots, \nu_p)$ of size $p = 1, \dots, r$, of the index set $\{1, 2, \dots, r\}$.

ASSUMPTION 5. The fourth moment $E \left[(X_{a_j}(t))^4 \right]$ is finite for all $a_j \in \{1, \dots, r\}$, while the fourth order cumulant function defined by

$$\text{cum} [X_{a_1}(t+t_1), X_{a_2}(t+t_2), X_{a_3}(t+t_3), X_{a_4}(t)]$$

does not depend on t , and this function, denoted by $C_{a_1 a_2 a_3 a_4}(t_1, t_2, t_3)$, satisfies

$$|C_{a_1 a_2 a_3 a_4}(t_1, t_2, t_3)| \leq \prod_{i=1}^3 g_{a_i}(t_i),$$

where $g_{a_i}(t_i)$, $i = 1, 2, 3$, are all continuous, even, nonnegative and integrable functions over the real line, which are non-increasing over $[0, \infty)$ for all $a_1, a_2, a_3, a_4 \in \{1, 2, \dots, r\}$.

Note that the cross spectral density is, in general, complex valued. Thus, the proposed estimator $\hat{\phi}_{a_1 a_2}(\cdot)$ can be represented as the vector

$$\begin{pmatrix} \text{Re} \left(\hat{\phi}_{a_1 a_2}(\lambda) \right) \\ \text{Im} \left(\hat{\phi}_{a_1 a_2}(\lambda) \right) \end{pmatrix}, \quad (3)$$

where

$$\begin{aligned}
& Re(\widehat{\phi}_{a_1 a_2}(\lambda)) \\
&= \frac{1}{2\pi n \rho_n} \sum_{t_1=1}^n \sum_{t_2=1}^n K(b_n(t_2 - t_1)) X_{a_1} \left(\frac{t_1}{\rho_n} \right) X_{a_2} \left(\frac{t_2}{\rho_n} \right) \cos \left(\frac{(t_2 - t_1)\lambda}{\rho_n} \right) 1_{[-\pi \rho_n, \pi \rho_n]}(\lambda), \\
& Im(\widehat{\phi}_{a_1 a_2}(\lambda)) \\
&= \frac{1}{2\pi n \rho_n} \sum_{t_1=1}^n \sum_{t_2=1}^n K(b_n(t_2 - t_1)) X_{a_1} \left(\frac{t_1}{\rho_n} \right) X_{a_2} \left(\frac{t_2}{\rho_n} \right) \sin \left(\frac{(t_2 - t_1)\lambda}{\rho_n} \right) 1_{[-\pi \rho_n, \pi \rho_n]}(\lambda).
\end{aligned}$$

THEOREM 2. *Under Assumptions 1–5, the covariance of $\begin{pmatrix} Re(\widehat{\phi}_{a_1 a_2}(\cdot)) \\ Im(\widehat{\phi}_{a_1 a_2}(\cdot)) \end{pmatrix}$ with $\begin{pmatrix} Re(\widehat{\phi}_{a_3 a_4}(\cdot)) \\ Im(\widehat{\phi}_{a_3 a_4}(\cdot)) \end{pmatrix}$*

converges as follows:

$$\lim_{n \rightarrow \infty} n b_n Cov \left[\begin{pmatrix} Re(\widehat{\phi}_{a_1 a_2}(\lambda_1)) \\ Im(\widehat{\phi}_{a_1 a_2}(\lambda_1)) \end{pmatrix}, \begin{pmatrix} Re(\widehat{\phi}_{a_3 a_4}(\lambda_2)) \\ Im(\widehat{\phi}_{a_3 a_4}(\lambda_2)) \end{pmatrix} \right] = \begin{bmatrix} \sigma_{11}(\lambda_1, \lambda_2) & \sigma_{12}(\lambda_1, \lambda_2) \\ \sigma_{21}(\lambda_1, \lambda_2) & \sigma_{22}(\lambda_1, \lambda_2) \end{bmatrix},$$

where

$$\begin{aligned}
\sigma_{11}(\lambda_1, \lambda_2) &= B \cdot Re \{ \phi_{a_1 a_3}^*(\lambda_2) \phi_{a_2 a_4}(\lambda_2) + \phi_{a_1 a_4}^*(\lambda_2) \phi_{a_2 a_3}(\lambda_2) \} \\
&\quad \times [1_{E_2}(\lambda_1, \lambda_2) + 1_{E_3}(\lambda_1, \lambda_2) + 2 \times 1_{E_4}(\lambda_1, \lambda_2)], \\
\sigma_{12}(\lambda_1, \lambda_2) &= B \cdot Im \{ \phi_{a_1 a_3}(\lambda_2) \phi_{a_2 a_4}^*(\lambda_2) + \phi_{a_1 a_4}^*(\lambda_2) \phi_{a_2 a_3}(\lambda_2) \} \\
&\quad \times [1_{E_2}(\lambda_1, \lambda_2) + 1_{E_3}(\lambda_1, \lambda_2)], \\
\sigma_{21}(\lambda_1, \lambda_2) &= B \cdot Im \{ \phi_{a_1 a_3}^*(\lambda_2) \phi_{a_2 a_4}(\lambda_2) + \phi_{a_1 a_4}^*(\lambda_2) \phi_{a_2 a_3}(\lambda_2) \} \\
&\quad \times [1_{E_2}(\lambda_1, \lambda_2) - 1_{E_3}(\lambda_1, \lambda_2)], \\
\sigma_{22}(\lambda_1, \lambda_2) &= B \cdot Re \{ \phi_{a_1 a_3}(\lambda_2) \phi_{a_2 a_4}^*(\lambda_2) - \phi_{a_1 a_4}(\lambda_2) \phi_{a_2 a_3}^*(\lambda_2) \} \\
&\quad \times [1_{E_2}(\lambda_1, \lambda_2) - 1_{E_3}(\lambda_1, \lambda_2)], \\
B &= \frac{1}{2} \int_{-\infty}^{\infty} K^2(x) dx, \\
E_1 &= \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 \neq 0, \lambda_1 + \lambda_2 \neq 0, -\infty < \lambda_1, \lambda_2 < \infty\}, \\
E_2 &= \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 = 0, -\infty < \lambda_1, \lambda_2 < \infty\} \setminus \{(0, 0)\}, \\
E_3 &= \{(\lambda_1, \lambda_2) : \lambda_1 + \lambda_2 = 0, -\infty < \lambda_1, \lambda_2 < \infty\} \setminus \{(0, 0)\}, \\
E_4 &= \{(0, 0)\}.
\end{aligned}$$

The convergence is uniform over any compact subset of E_1 , E_2 or E_3 . In particular, the variance-

covariance matrix of the random vector $\begin{pmatrix} Re(\widehat{\phi}_{a_1 a_2}(\cdot)) \\ Im(\widehat{\phi}_{a_1 a_2}(\cdot)) \end{pmatrix}$ goes to zero as $n \rightarrow \infty$, for all $a_1, a_2 \in \{1, 2, \dots, r\}$.

The covariance between two complex-valued random variables is often defined as the trace of the 2×2 cross-covariance matrix of the random vectors formed by their real and imaginary parts (Brockwell and Davis, 1991). In the case of the pair $(\widehat{\phi}_{a_1 a_2}(\lambda_1), \widehat{\phi}_{a_3 a_4}(\lambda_2))$, the limiting covariance according to this notion can be easily be computed from Theorem 2.

Theorem 1 and Theorem 2 together establish the consistency of any vector of estimators having elements of the form $\widehat{\phi}_{a_1 a_2}(\cdot)$.

3 Asymptotic Normality

We will make an additional assumption about the underlying process in order to prove the asymptotic normality of the estimator.

ASSUMPTION 5A. The process \mathbf{X} is strictly stationary; all moments of the process exist, i.e., $E[(X_a(t))^k] < \infty$ for each $k > 2$ and for all $a \in \{1, \dots, r\}$; and for each $a_1, a_2, \dots, a_k \in \{1, 2, \dots, r\}$ and each $k > 2$, the k th order joint cumulant denoted by

$$C_{a_1 a_2 \dots a_k}(t_1, t_2, \dots, t_{k-1}) = \text{cum}(X_{a_1}(t_1 + t), X_{a_2}(t_2 + t), \dots, X_{a_{k-1}}(t_{k-1} + t), X_{a_k}(t)),$$

satisfies

$$|C_{a_1 a_2 \dots a_k}(t_1, t_2, \dots, t_{k-1})| \leq \prod_{i=1}^{k-1} g_{a_i}(t_i),$$

where $g_{a_i}(t_i)$, $i = 1, \dots, k-1$ are continuous, even, nonnegative and integrable functions over the real line, which are non-increasing over $(0, \infty)$.

Note that Assumption 5A is stronger than Assumption 5.

The following theorem describes the asymptotic behaviour of the joint cumulants of the estimators $\widehat{\phi}_{a_1 a_2}(\cdot)$ for $a_1, a_2 \in \{1, 2, \dots, r\}$. In the present case, a cumulant defined as in (2) may be complex-valued.

THEOREM 3. *Under the Assumptions 1–4 and 5A, for $L > 2$, the L th order joint cumulant of the vector $(\widehat{\phi}_{a_1 a_2}(\lambda_1), \dots, \widehat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L))$ for $a_1, a_2, \dots, a_{2L} \in \{1, 2, \dots, r\}$ is bounded from above as follows.*

$$\left| \text{cum}(\widehat{\phi}_{a_1 a_2}(\lambda_1), \dots, \widehat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L)) \right| \leq Q \cdot (nb_n)^{-(L-1)}, \quad (4)$$

where the constant Q does not depend on $\lambda_1, \dots, \lambda_L$.

THEOREM 4. Under Assumptions 1–4 and 5A, a vector of real and imaginary parts of estimated spectra or cross-spectra converges weakly as follows.

$$\sqrt{nb_n} \begin{bmatrix} \begin{pmatrix} \operatorname{Re}\{\hat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \operatorname{Im}\{\hat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \vdots \\ \operatorname{Re}\{\hat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \\ \operatorname{Im}\{\hat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \end{pmatrix} - E \begin{pmatrix} \operatorname{Re}\{\hat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \operatorname{Im}\{\hat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \vdots \\ \operatorname{Re}\{\hat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \\ \operatorname{Im}\{\hat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \end{pmatrix} \end{bmatrix} \xrightarrow{D} N_{2J}(0, \Sigma), \quad (5)$$

where $a_1, a_2, \dots, a_{2J} \in \{1, 2, \dots, r\}$, and the elements of Σ are defined in accordance with Theorem 2.

The foregoing theorem only shows that the vector estimator, after appropriate mean adjustment and scaling, converges weakly to a multivariate normal distribution. However, weak convergence around the true vector of spectra and cross-spectra remains to be established. Note that

$$\begin{aligned} \sqrt{nb_n} \left(\hat{\phi}_{a_1 a_2}(\lambda) - \phi_{a_1 a_2}(\lambda) \right) &= \sqrt{nb_n} \left(\hat{\phi}_{a_1 a_2}(\lambda) - E[\hat{\phi}_{a_1 a_2}(\lambda)] \right) \\ &\quad + \sqrt{nb_n} \left(E[\hat{\phi}_{a_1 a_2}(\lambda)] - \phi_{a_1 a_2}(\lambda) \right). \end{aligned} \quad (6)$$

We make some further assumptions on the smoothness and the rate of decay of the spectrum and the shape of the kernel function in order to obtain the rate of convergence of the bias $E[\hat{\phi}_{a_1 a_2}(\lambda)] - \phi_{a_1 a_2}(\lambda)$.

ASSUMPTION 1A. The function $g_{qa_1 a_2}(\cdot)$, defined over the real line as

$$g_{qa_1 a_2}(t) = \sup_{|s| \geq |t|} |s|^q |C_{a_1 a_2}(s)|$$

is integrable for all $a_1, a_2 \in \{1, 2, \dots, r\}$, for some positive number q greater than 1.

ASSUMPTION 1B. The power spectral density is such that, for all $a_1, a_2 \in \{1, 2, \dots, r\}$ and for some $p > 1$, $\lim_{\lambda \rightarrow \infty} |\lambda^p \phi_{a_1 a_2}(\lambda)| = A_{a_1 a_2}$ for some non-negative number $A_{a_1 a_2}$.

For any kernel $K(\cdot)$, let us define

$$k_s = \lim_{x \rightarrow 0} \frac{1 - K(x)}{|x|^s}$$

for each positive number s such that the limit exists. The characteristic exponent of the kernel is defined as the largest number s , such that the limit exists and is non-zero (Parzen, 1957). In other words, the characteristic exponent is the number s such that $1 - K(1/y)$ is $O(y^{-s})$.

ASSUMPTION 2A. The characteristic exponent of the kernel $K(\cdot)$ is a number, for which Assumption 1A holds.

Note that Assumption 1A implies Assumption 1, and also that $\phi_{a_1 a_2}(\cdot)$ is $[q]$ times differentiable, where $[q]$ is the integer part of q . Thus, the number q indicates the degree of smoothness of the spectral density. If Assumption 1A holds for a particular value of q , then it would also hold for smaller values.

The number p indicates the slowest rate of decay of the various elements of the power spectral density matrix. The following are two interesting situations, where Assumption 1B holds.

1. The real and imaginary parts of the components of the power spectral density matrix are rational functions of the form $\frac{P(\lambda)}{Q(\lambda)}$, where $P(\cdot)$ and $Q(\cdot)$ are polynomials such that the degree of $Q(\cdot)$ is more than degree of $P(\cdot)$ by at least p . Note that continuous time ARMA processes possess rational power spectral density.
2. The function $C_{a_1 a_2}(\cdot)$ has the following smoothness property: $C_{a_1 a_2}(\cdot)$ is p times differentiable and the p^{th} derivative of $C_{a_1 a_2}(\cdot)$ is in L^1 .

THEOREM 5. Under Assumptions 2–4, 1A, 1B and 2A, the bias of the estimator $\hat{\phi}_{a_1 a_2}(\lambda)$ given by (1), for $a_1, a_2 \in \{1, 2, \dots, r\}$, is

$$\begin{aligned} E[\hat{\phi}_{a_1 a_2}(\lambda) - \phi_{a_1 a_2}(\lambda)] &= \left[-\frac{k_q}{2\pi} \int_{-\infty}^{\infty} |t|^q C_{a_1 a_2}(t) e^{-it\lambda} dt \right] (\rho_n b_n)^q + o((\rho_n b_n)^q) \\ &\quad + \left[-\frac{1}{2\pi} \int_{-\infty}^{\infty} |t| C_{a_1 a_2}(t) e^{-it\lambda} dt \right] \left(\frac{\rho_n}{n} \right) + o\left(\frac{\rho_n}{n} \right) \\ &\quad + \left[\frac{A_{a_1 a_2}}{(2\pi)^p} \sum_{|l|>0} \frac{1}{|l|^p} \right] \frac{1}{(\rho_n)^p} + o\left(\frac{1}{(\rho_n)^p} \right). \end{aligned}$$

Theorem 5 shows that the second term in (6) would go to zero if the sampling rate ρ_n satisfies additional conditions.

ASSUMPTION 4A. The sampling rate is such that $\sqrt{nb_n}(\rho_n b_n)^q \rightarrow 0$ and $\sqrt{nb_n}/\rho_n^p \rightarrow 0$ as $n \rightarrow \infty$.

Note that, whenever Assumption 3 holds, Assumption 4A is stronger than Assumption 4. With this assumption, the expected values of the estimators in Theorem 4 can be replaced by the respective true values.

THEOREM 6. *Under Assumptions 1–3, 1A, 1B, 2A, 4A and 5A, we have the following weak convergence.*

$$\sqrt{nb_n} \left[\begin{pmatrix} \operatorname{Re}\{\widehat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \operatorname{Im}\{\widehat{\phi}_{a_1 a_2}(\lambda_1)\} \\ \vdots \\ \operatorname{Re}\{\widehat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \\ \operatorname{Im}\{\widehat{\phi}_{a_{2J-1} a_{2J}}(\lambda_J)\} \end{pmatrix} - \begin{pmatrix} \operatorname{Re}\{\phi_{a_1 a_2}(\lambda_1)\} \\ \operatorname{Im}\{\phi_{a_1 a_2}(\lambda_1)\} \\ \vdots \\ \operatorname{Re}\{\phi_{a_{2J-1} a_{2J}}(\lambda_J)\} \\ \operatorname{Im}\{\phi_{a_{2J-1} a_{2J}}(\lambda_J)\} \end{pmatrix} \right] \xrightarrow{D} N_{2J}(0, \Sigma),$$

where $a_1, a_2, \dots, a_{2J} \in \{1, 2, \dots, r\}$, and the elements of Σ are defined in accordance with Theorem 2.

4 Optimal rate of convergence

We are now in a position to optimize the rates of b_n and ρ_n so that $\frac{1}{\sqrt{nb_n}}$ tends to 0 as fast as possible under the conditions of Theorem 6.

THEOREM 7. *Under Assumptions 3 and 4A, the reciprocal of the scale factor ($\frac{1}{\sqrt{nb_n}}$) used in Theorem 6 has the fastest convergence to 0 when*

$$\begin{aligned} b_n &= o\left(n^{-\frac{p+q}{p+q+2pq}}\right), \\ \rho_n &= O\left(n^{\frac{q}{p+q+2pq}}\right), \end{aligned}$$

and under these conditions, $\frac{1}{\sqrt{nb_n}} = o\left(n^{-\frac{pq}{p+q+2pq}}\right)$.

It has been shown in Srivastava and Sengupta (2010) that under the assumptions of Theorems 2 and 5, the optimal rate of convergence for mean square consistency of the estimator (1) is given as

$$E \left[\{\widehat{\phi}_{a_1 a_2}(\cdot) - \phi_{a_1 a_2}(\cdot)\}^2 \right] = O\left(n^{-\frac{2pq}{p+q+2pq}}\right),$$

which corresponds to the choices

$$\begin{aligned} b_n &= O\left(n^{-\frac{p+q}{p+q+2pq}}\right), \\ \rho_n &= O\left(n^{\frac{q}{p+q+2pq}}\right). \end{aligned}$$

Theorem 7 shows that the optimal rate of weak convergence of the estimator $\widehat{\phi}_{a_1 a_2}(\cdot)$ is slower than the square root of the optimal rate corresponding to mean square consistency.

It is important to note that for every fixed value of q , the number p , which indicates rate of decay of the spectrum, can be increased indefinitely by continuous time low pass filtering with a cut off

frequency larger than the maximum frequency of interest. There are well-known filters such as the Butterworth filter, which have polynomial rate of decay of the transfer function with specified degree of the polynomial, that can be used for this purpose. For fixed q , the best rate of weak convergence given in Theorem 7, obtained by allowing p to go to infinity, happens to be $o\left(n^{-\frac{q}{1+2q}}\right)$.

The rate of weak convergence crucially depends on the number q , the assumed degree of smoothness of the spectrum. The stronger the assumption, the faster is the rate of convergence. The rate corresponding to $q = 1$ (weakest possible assumption) is $o\left(n^{-\frac{1}{3}}\right)$, assuming that p can be allowed to be very large. For very large q (very strong assumption) and large p , the rate approaches $o\left(n^{-\frac{1}{2}}\right)$.

5 Simulation

With a view to investigating the applicability of the asymptotic results reported in Section 3 to finite sample size, we consider the bivariate continuous time linear process

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^t h_1(t-u)Z_1(u)du + \int_{-\infty}^t h_2(t-u)Z_2(u)du \\ \int_{-\infty}^t h_3(t-u)Z_1(u)du + \int_{-\infty}^t h_4(t-u)Z_3(u)du \end{pmatrix}$$

where $Z_j(u)$, $j = 1, 2, 3$ are independent continuous time white noise and $h_j(u) = \beta_j e^{-\alpha_j u}$ for $j \in \{1, 2, 3, 4\}$. The elements of the spectral density matrix

$$\begin{pmatrix} \phi_{11}(\lambda) & \phi_{12}(\lambda) \\ \phi_{12}^*(\lambda) & \phi_{22}(\lambda) \end{pmatrix}$$

are defined as follows (Hoel et al., 1972).

$$\begin{aligned} \phi_{11}(\lambda) &= \frac{1}{2\pi} \cdot \frac{\beta_1^2}{\alpha_1^2 + \lambda^2} + \frac{1}{2\pi} \cdot \frac{\beta_2^2}{\alpha_2^2 + \lambda^2}, \\ \phi_{22}(\lambda) &= \frac{1}{2\pi} \cdot \frac{\beta_3^2}{\alpha_3^2 + \lambda^2} + \frac{1}{2\pi} \cdot \frac{\beta_4^2}{\alpha_4^2 + \lambda^2}, \\ \text{Re}(\phi_{12}(\lambda)) &= \frac{1}{2\pi} \cdot \frac{\beta_1 \beta_3 (\alpha_1 \alpha_3 + \lambda^2)}{(\alpha_1^2 + \lambda^2)(\alpha_3^2 + \lambda^2)} \\ \text{and } \text{Im}(\phi_{12}(\lambda)) &= \frac{1}{2\pi} \cdot \frac{\beta_1 \beta_3 (\alpha_3 - \alpha_1) \lambda}{(\alpha_1^2 + \lambda^2)(\alpha_3^2 + \lambda^2)}. \end{aligned}$$

We simulate this bivariate process with the choices $\beta_1 = 1, \beta_2 = 1, \beta_3 = 2, \beta_4 = \frac{2}{5}, \alpha_1 = \beta_1 \cdot \sqrt{\frac{3}{2}}, \alpha_2 = \beta_2 \cdot \sqrt{3}, \alpha_3 = \beta_3 \cdot \sqrt{3}$ and $\alpha_4 = \beta_4 \cdot \sqrt{3}$. Note that for this process, Assumption 1A holds with $q \geq 1$ and Assumption 1B holds with $p \leq 2$. For the purpose of estimation, we make these assumptions with $p = 2$ and $q = 2$. In accordance with this choice of q , we use the second order

kernel function

$$K(x) = \frac{1}{2} \{1 + \cos(\pi x)\} 1_{[-1,1]}(x).$$

We also use the rates $b_n = \frac{1}{4}n^{-\frac{1}{4}}$ and $\rho_n = 4 \cdot n^{\frac{1}{6}}$.

We estimate the bivariate spectrum matrix for frequencies in the range $[0, 3\pi]$ at intervals of $.01\pi$ (i.e., 301 uniformly spaced grid points). We subsequently compute the normalized statistics

$$\begin{aligned} T_1(\lambda) &= \sqrt{nb_n} \left(\frac{\hat{\phi}_{11}(\lambda) - \phi_{11}(\lambda)}{\sqrt{2\{1 + 1_{\{0\}}(\lambda)\}B\hat{\phi}_{11}^2(\lambda)}} \right), \\ T_2(\lambda) &= \sqrt{nb_n} \left(\frac{\hat{\phi}_{22}(\lambda) - \phi_{22}(\lambda)}{\sqrt{2\{1 + 1_{\{0\}}(\lambda)\}B\hat{\phi}_{22}^2(\lambda)}} \right), \\ T_3(\lambda) &= \sqrt{nb_n} \left(\frac{Re(\hat{\phi}_{12}(\lambda)) - Re(\phi_{12}(\lambda))}{\sqrt{\{1 + 1_{\{0\}}(\lambda)\}B[\hat{\phi}_{11}(\lambda)\hat{\phi}_{22}(\lambda) + \{Re(\hat{\phi}_{12}(\lambda))\}^2 - \{Im(\hat{\phi}_{12}(\lambda))\}^2]}} \right), \\ T_4(\lambda) &= \sqrt{nb_n} \left(\frac{Im(\hat{\phi}_{12}(\lambda)) - Im(\phi_{12}(\lambda))}{\sqrt{B[\hat{\phi}_{11}(\lambda)\hat{\phi}_{22}(\lambda) - \{Re(\hat{\phi}_{12}(\lambda))\}^2 + \{Im(\hat{\phi}_{12}(\lambda))\}^2]}} \right) [1 - 1_{\{0\}}(\lambda)], \end{aligned}$$

in accordance with Theorem 2. According to Theorem 6, the asymptotic distribution of each of these four statistics is standard normal. This procedure is repeated for 500 simulation runs. By regading the values of the above statistics for the different simulation runs as four data sets of size 500 each, we calculate the Kolmogorov-Smirnov test statistic (Shorak and Wellner, 1986) for these data sets, and the corresponding p-value. This procedure is repeated for the 301 frequency values mentioned above. The percentage of p-values (across 301 frequency values) exceeding the number 0.05 are reported in Table 1, for sample sizes $n = 100, 1000, 10000$ and 100000 . The table shows that for each statistic, the percentage approaches the ideal value of 95 very slowly as n increases.

We now turn to computation of confidence limits of the power spectral density. For each frequency value, we compute the 95% asymptotic confidence intervals of ϕ_{11} , ϕ_{22} , $Re(\phi_{12})$ and $Im(\phi_{12})$ from the statistics $T_1(\lambda)$, $T_2(\lambda)$, $T_3(\lambda)$ and $T_4(\lambda)$, assuming that the latter have the standard normal distribution. Subsequently, we compute the fraction of times (out of 500 simulation runs) the confidence limits capture the true value of the function. These percentages are plotted against the frequency, for sample sizes $n = 100, 1000, 10000$ and 100000 , in Figure 1. It is seen that the observed fraction approaches the ideal coverage probability (0.95) for larger sample sizes. Since there is a discontinuity of the asymptotic variance function at the point $\lambda = 0$, while the estimated spectrum is constrained to be continuous, some anomalous behaviour in the neighbourhood

sample size (n)	observed percentage			
	ϕ_{11}	ϕ_{22}	$Re(\phi_{12})$	$Im(\phi_{12})$
100	0.0 %	0.0 %	0.0 %	0.0 %
1000	4.3 %	6.0 %	9.6 %	16.3 %
10000	73.4 %	72.4 %	76.1 %	72.8 %
100000	91.4 %	88.4 %	93.4 %	90.0 %

Table 1. Observed percentage of frequencies (in the range 0 to 3π) for which p-values of the Kolmogorov-Smirnov statistics for testing normality of ϕ_{11} , ϕ_{22} , $Re(\phi_{12})$ and $Im(\phi_{12})$ are greater than 0.05 (ideal percentage is 95%).

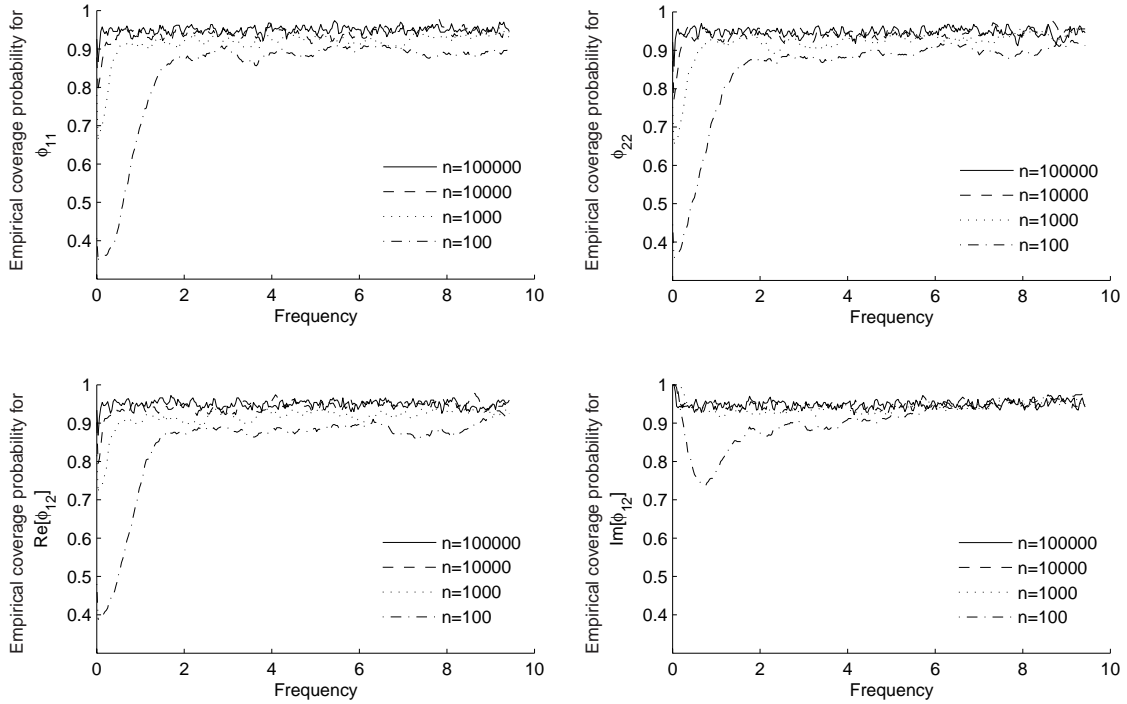


Figure 1. Empirical coverage probability (based on 500 simulation runs) of pointwise confidence intervals of ϕ_{11} , ϕ_{22} , $Re(\phi_{12})$ and $Im(\phi_{12})$ for sample sizes 100, 1000, 10000 and 100000.

of the the point $\lambda = 0$ is expected. This results in substantially lower values of the empirical coverage probability in this region. However, this region of anomaly is observed to shrink as the sample size increases. It would be interesting to note that the empirical coverage probability is reasonably close to the ideal coverage probability for most frequency values when the sample size as small as 1000, even though Table 1 indicates that the asymptotic distribution is not applicable at this sample size.

Appendix

We denote by $K_1(\cdot)$ a function that bounds the covariance averaging kernel $K(\cdot)$ as in Assumption 2.

Further, we denote $K_1(0)$ by M .

PROOF OF THEOREM 1. We shall show that the bias of the estimator $\widehat{\phi}_{a_1 a_2}(\lambda)$ given by (1) converges to 0 uniformly over $[\lambda_l, \lambda_u]$ for any λ_l, λ_u such that $\lambda_l < \lambda_u$. Note that

$$E[\widehat{\phi}_{a_1 a_2}(\lambda)] = \frac{1}{2\pi\rho_n} \sum_{u=-(n-1)}^{n-1} \left(1 - \frac{|u|}{\rho_n}\right) K(b_n u) C_{a_1 a_2} \left(\frac{u}{\rho_n}\right) e^{-\frac{i u \lambda}{\rho_n}} 1_{[-\pi\rho_n, \pi\rho_n]}(\lambda).$$

Consider the simple function $S_n(\cdot)$, defined over $[\lambda_l, \lambda_u] \times \mathbb{R}$, by

$$S_n(\lambda, x) = \frac{1}{2\pi} \sum_{u=-(n-1)}^{n-1} \left(1 - \frac{|u|}{\rho_n}\right) K(b_n u) C_{a_1 a_2} \left(\frac{u}{\rho_n}\right) e^{-\frac{i u \lambda}{\rho_n}} 1_{[-\pi\rho_n, \pi\rho_n]}(\lambda) 1_{\left(\frac{u-1}{\rho_n}, \frac{u}{\rho_n}\right]}(x).$$

Observe that $\int_{-\infty}^{\infty} S_n(\lambda, x) dx = E[\widehat{\phi}_{a_1 a_2}(\lambda)]$. Define the function $S(\cdot)$, over $[\lambda_l, \lambda_u] \times \mathbb{R}$, by $S(\lambda, x) = \frac{1}{2\pi} C_{a_1 a_2}(x) e^{-i x \lambda}$.

For any $x \in \mathbb{R}$, let $u_n(x)$ be the smallest integer greater than or equal to $\rho_n x$. Note that the interval $\left(\frac{u_n-1}{\rho_n}, \frac{u_n}{\rho_n}\right]$ contains the point x and $\lim_{n \rightarrow \infty} \frac{u_n(x)}{\rho_n} = x$. For sufficiently large n , we have from Assumptions 3 and 4,

$$S_n(\lambda, x) = \frac{1}{2\pi} \left(1 - \frac{|u_n(x)|}{\rho_n} \frac{\rho_n}{n}\right) K\left(b_n \rho_n \frac{u_n(x)}{\rho_n}\right) C_{a_1 a_2} \left(\frac{u_n(x)}{\rho_n}\right) e^{-\frac{i u_n(x) \lambda}{\rho_n}} 1_{[-\pi\rho_n, \pi\rho_n]}(\lambda).$$

Proving the uniform convergence of $Bias[\widehat{\phi}_{a_1 a_2}(\lambda)]$ over the finite interval $[\lambda_l, \lambda_u]$ amounts to proving

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} S_n(\lambda, x) dx = \int_{-\infty}^{\infty} S(\lambda, x) dx, \quad (\text{A.1})$$

uniformly over $[\lambda_l, \lambda_u]$.

Observe that $\int_{-\infty}^{\infty} S(\lambda, t) dt = \phi_{a_1 a_2}(\lambda)$, which is continuous. By virtue of the continuity of the limiting function, (A.1) is equivalent to proving that $\int_{-\infty}^{\infty} S_n(\lambda, x) dx$ converges continuously over this interval (Resnick, 1987), i.e., for any sequence $\lambda_n \rightarrow \lambda$,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} S_n(\lambda_n, x) dx = \int_{-\infty}^{\infty} S(\lambda, x) dx, \quad (\text{A.2})$$

where $\lambda_n, \lambda \in [\lambda_l, \lambda_u]$.

By continuity of the function $S_n(\lambda, x)$ with respect to x and λ , we have from Assumptions 3 and 4, for any fixed x ,

$$\lim_{n \rightarrow \infty} |S_n(\lambda_n, x) - S(\lambda, x)| = 0.$$

Note that from Assumptions 1 and 2, we have the dominance

$$|S_n(\lambda_n, x)| \leq M \sum_{|u| < n} \left| C_{a_1 a_2} \left(\frac{u}{\rho_n} \right) \right| 1_{\left(\frac{u-1}{\rho_n}, \frac{u}{\rho_n} \right]}(x) \leq M g_{a_1 a_2}(x),$$

where $g_{a_1 a_2}(\cdot)$ is the function described in Assumption 1. Thus, by applying the dominated convergence theorem (DCT), we have (A.2).

Hence, $E[\widehat{\phi}_{a_1 a_2}(\lambda)] \rightarrow \phi(\lambda)$ uniformly on $[\lambda_l, \lambda_u]$. \square

PROOF OF THEOREM 2. We begin by calculating the covariance between the estimators $Re(\widehat{\phi}_{a_1 a_2}(\cdot))$ and $Re(\widehat{\phi}_{a_3 a_4}(\cdot))$.

$$\begin{aligned} & Cov \left[Re(\widehat{\phi}_{a_1 a_2}(\lambda_1)), Re(\widehat{\phi}_{a_3 a_4}(\lambda_2)) \right] \\ &= \frac{1}{(2\pi)^2 (n\rho_n)^2} \sum_{t_1=1}^n \sum_{t_2=1}^n \sum_{t_3=1}^n \sum_{t_4=1}^n K(b_n(t_2 - t_1)) K(b_n(t_4 - t_3)) \\ & \quad \times Cov \left[X_{a_1} \left(\frac{t_1}{\rho_n} \right) X_{a_2} \left(\frac{t_2}{\rho_n} \right), X_{a_3} \left(\frac{t_3}{\rho_n} \right) X_{a_4} \left(\frac{t_4}{\rho_n} \right) \right] \cos \left(\frac{(t_2 - t_1)\lambda_1}{\rho_n} \right) \cos \left(\frac{(t_4 - t_3)\lambda_2}{\rho_n} \right) \\ &= \frac{1}{(2\pi)^2 (n\rho_n)^2} \sum_{t_1=1}^n \sum_{t_2=1}^n \sum_{t_3=1}^n \sum_{t_4=1}^n K(b_n(t_1 - t_2)) K(b_n(t_3 - t_4)) \\ & \quad \times \left[C_{a_1 a_3} \left(\frac{t_1 - t_3}{\rho_n} \right) C_{a_2 a_4} \left(\frac{t_2 - t_4}{\rho_n} \right) + C_{a_1 a_4} \left(\frac{t_1 - t_4}{\rho_n} \right) C_{a_2 a_3} \left(\frac{t_2 - t_3}{\rho_n} \right) \right. \\ & \quad \left. + C_{a_1 a_2 a_3 a_4} \left(\frac{t_1 - t_4}{\rho_n}, \frac{t_2 - t_4}{\rho_n}, \frac{t_3 - t_4}{\rho_n} \right) \right] \cos \left(\frac{(t_1 - t_2)\lambda_1}{\rho_n} \right) \cos \left(\frac{(t_3 - t_4)\lambda_2}{\rho_n} \right) \\ &= T_1(\lambda_1, \lambda_2) + T_2(\lambda_1, \lambda_2) + T_3(\lambda_1, \lambda_2), \end{aligned}$$

where the three terms correspond to the three summands appearing inside square brackets in the previous step.

Now consider the function $T_1(\lambda_1, \lambda_2)$. By using the transformations $u_1 = t_1 - t_2$, $u_2 = t_1 - t_3$ and $u_3 = t_2 - t_4$, we have

$$\begin{aligned} T_1(\lambda_1, \lambda_2) &= \frac{1}{(2\pi)^2 (n\rho_n)^2} \sum_{t_1=1}^n \sum_{u_1=t_1-1}^{t_1-n} \sum_{u_2=t_1-1}^{n-t_1} \sum_{u_3=t_1-1-u_1}^{t_1-n-u_1} K(b_n u_1) K(b_n(u_1 - u_2 + u_3)) \\ & \quad \times C_{a_1 a_3} \left(\frac{u_2}{\rho_n} \right) C_{a_2 a_4} \left(\frac{u_3}{\rho_n} \right) \cos \left(\frac{u_1 \lambda_1}{\rho_n} \right) \cos \left(\frac{(u_1 - u_2 + u_3) \lambda_2}{\rho_n} \right). \end{aligned}$$

The range of the four summations on the right hand side is described by the set of inequalities $1 \leq t_1 \leq n$ and $t_1 - n \leq u_1, u_2, u_1 + u_3 \leq t_1 - 1$, which is equivalent to the inequalities $-(n-1) \leq u_1, u_2, u_1 + u_3 \leq (n-1)$ and $\max\{u_1, u_2, u_1 + u_3\} + 1 \leq t_1 \leq \min\{u_1, u_2, u_1 + u_3\}$.

Therefore, the expression for $T_1(\lambda_1, \lambda_2)$ simplifies to

$$\begin{aligned} & \frac{1}{(2\pi)^2 n \rho_n^2} \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(u_1 - u_2 + u_3)) \\ & \times C_{a_1 a_3} \left(\frac{u_2}{\rho_n} \right) C_{a_2 a_4} \left(\frac{u_3}{\rho_n} \right) \cos \left(\frac{u_1 \lambda_1}{\rho_n} \right) \cos \left(\frac{(u_1 - u_2 + u_3) \lambda_2}{\rho_n} \right), \end{aligned}$$

where

$$U_n(u_1, u_2, u_3) = \left(1 + \frac{\min(u_1, u_2, u_1 + u_3)}{n} - \frac{\max(u_1, u_2, u_1 + u_3)}{n} \right).$$

By writing the cosine functions in terms of complex exponentials, we have

$$\begin{aligned} & T_1(\lambda_1, \lambda_2) \\ &= \frac{1}{(4\pi)^2 n \rho_n^2} \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(u_1 - u_2 + u_3)) \\ & \times C_{a_1 a_3} \left(\frac{u_2}{\rho_n} \right) C_{a_2 a_4} \left(\frac{u_3}{\rho_n} \right) \left\{ e^{-i \frac{(\lambda_1 - \lambda_2) u_1}{\rho_n}} e^{-i \frac{\lambda_2 u_2}{\rho_n}} e^{i \frac{\lambda_2 u_3}{\rho_n}} + e^{i \frac{(\lambda_1 - \lambda_2) u_1}{\rho_n}} e^{i \frac{\lambda_2 u_2}{\rho_n}} e^{-i \frac{\lambda_2 u_3}{\rho_n}} \right. \\ & \quad \left. + e^{i \frac{(\lambda_1 + \lambda_2) u_1}{\rho_n}} e^{-i \frac{\lambda_2 u_2}{\rho_n}} e^{i \frac{\lambda_2 u_3}{\rho_n}} + e^{-i \frac{(\lambda_1 + \lambda_2) u_1}{\rho_n}} e^{i \frac{\lambda_2 u_2}{\rho_n}} e^{-i \frac{\lambda_2 u_3}{\rho_n}} \right\} \\ &= T_{11}(\lambda_1, \lambda_2) + T_{12}(\lambda_1, \lambda_2) + T_{13}(\lambda_1, \lambda_2) + T_{14}(\lambda_1, \lambda_2), \end{aligned} \tag{A.3}$$

where the four terms correspond to the four summands appearing within braces in the last factor on the right hand side of (A.3).

By using the results of Lemmas 1 and 2 given below, we have the convergence

$$\lim_{n \rightarrow \infty} n b_n T_{11}(\lambda_1, \lambda_2) = \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_3}(\lambda_2) \phi_{a_2 a_4}^*(\lambda_2) 1_{E_2 \cup E_4}(\lambda_1, \lambda_2)$$

and similar arguments show that

$$\begin{aligned} \lim_{n \rightarrow \infty} n b_n T_{12}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_3}^*(\lambda_2) \phi_{a_2 a_4}(\lambda_2) 1_{E_2 \cup E_4}(\lambda_1, \lambda_2), \\ \lim_{n \rightarrow \infty} n b_n T_{13}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_3}(\lambda_2) \phi_{a_2 a_4}^*(\lambda_2) 1_{E_3 \cup E_4}(\lambda_1, \lambda_2) \text{ and} \\ \lim_{n \rightarrow \infty} n b_n T_{14}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_3}^*(\lambda_2) \phi_{a_2 a_4}(\lambda_2) 1_{E_3 \cup E_4}(\lambda_1, \lambda_2). \end{aligned}$$

For the function $T_2(\lambda_1, \lambda_2)$, one can similarly use the transformations $u_1 = t_1 - t_2$, $u_2 = t_1 - t_4$ and $u_3 = t_2 - t_3$, interchange the order of summation and expand the cosine functions in terms of

complex exponentials to obtain

$$\begin{aligned}
& T_2(\lambda_1, \lambda_2) \\
&= \frac{1}{(2\pi)^2 n \rho_n^2} \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(-u_1 + u_2 - u_3)) \\
&\quad \times C_{a_1 a_4} \left(\frac{u_2}{\rho_n} \right) C_{a_2 a_3} \left(\frac{u_3}{\rho_n} \right) \cos \left(\frac{u_1 \lambda_1}{\rho_n} \right) \cos \left(\frac{(-u_1 + u_2 - u_3) \lambda_2}{\rho_n} \right) \\
&= \frac{1}{(4\pi)^2 n \rho_n^2} \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(-u_1 + u_2 - u_3)) \\
&\quad \times C_{a_1 a_4} \left(\frac{u_2}{\rho_n} \right) C_{a_2 a_3} \left(\frac{u_3}{\rho_n} \right) \left\{ e^{-i \frac{(\lambda_1 - \lambda_2) u_1}{\rho_n}} e^{-i \frac{\lambda_2 u_2}{\rho_n}} e^{i \frac{\lambda_2 u_3}{\rho_n}} + e^{i \frac{(\lambda_1 - \lambda_2) u_1}{\rho_n}} e^{i \frac{\lambda_2 u_2}{\rho_n}} e^{-i \frac{\lambda_2 u_3}{\rho_n}} \right. \\
&\quad \left. + e^{i \frac{(\lambda_1 + \lambda_2) u_1}{\rho_n}} e^{-i \frac{\lambda_2 u_2}{\rho_n}} e^{i \frac{\lambda_2 u_3}{\rho_n}} + e^{-i \frac{(\lambda_1 + \lambda_2) u_1}{\rho_n}} e^{i \frac{\lambda_2 u_2}{\rho_n}} e^{-i \frac{\lambda_2 u_3}{\rho_n}} \right\} \\
&= T_{21}(\lambda_1, \lambda_2) + T_{22}(\lambda_1, \lambda_2) + T_{23}(\lambda_1, \lambda_2) + T_{24}(\lambda_1, \lambda_2).
\end{aligned}$$

By using similar arguments as in the case of $nb_n T_{11}(\lambda_1, \lambda_2)$, it can be shown that

$$\begin{aligned}
nb_n T_{21}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_4}(\lambda_2) \phi_{a_2 a_3}^*(\lambda_2) 1_{E_2 \cup E_4}(\lambda_1, \lambda_2), \\
nb_n T_{22}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_4}^*(\lambda_2) \phi_{a_2 a_3}(\lambda_2) 1_{E_2 \cup E_4}(\lambda_1, \lambda_2), \\
nb_n T_{23}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_4}(\lambda_2) \phi_{a_2 a_3}^*(\lambda_2) 1_{E_3 \cup E_4}(\lambda_1, \lambda_2) \text{ and} \\
nb_n T_{24}(\lambda_1, \lambda_2) &= \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_4}^*(\lambda_2) \phi_{a_2 a_3}(\lambda_2) 1_{E_3 \cup E_4}(\lambda_1, \lambda_2).
\end{aligned}$$

Finally, for the term $T_3(\lambda_1, \lambda_2)$, we use the transformations $u_1 = t_1 - t_4$, $u_2 = t_2 - t_4$ and $u_3 = t_3 - t_4$ and interchange the order of summations to have

$$\begin{aligned}
T_3(\lambda_1, \lambda_2) &= \frac{1}{(2\pi)^2 (n \rho_n)^2} \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{n-1} \sum_{u_3=-(n-1)}^{(n-1)} (n - \min(u_1, u_2, u_3) + \max(u_1, u_2, u_3)) \\
&\quad K(b_n(u_1 - u_2)) K(b_n u_3) C_{a_1 a_2 a_3 a_4} \left(\frac{u_1}{\rho_n}, \frac{u_2}{\rho_n}, \frac{u_3}{\rho_n} \right) \cos \left(\frac{(u_1 - u_2) \lambda_1}{\rho_n} \right) \cos \left(\frac{u_3 \lambda_2}{\rho_n} \right).
\end{aligned}$$

From Assumptions 2 and 5, we have

$$nb_n |T_3(\lambda_1, \lambda_2)| \leq \rho_n b_n M^2 \sum_{u_1=-(n-1)}^{n-1} \sum_{u_2=-(n-1)}^{n-1} \sum_{u_3=-(n-1)}^{n-1} g_{a_1} \left(\frac{u_1}{\rho_n} \right) g_{a_2} \left(\frac{u_2}{\rho_n} \right) g_{a_3} \left(\frac{u_3}{\rho_n} \right) \frac{1}{\rho_n^3}. \quad (\text{A.4})$$

Now consider the function $S_n(\cdot)$ defined over \mathbb{R} as

$$S_n(x) = \sum_{u_1=-(n-1)}^{n-1} g_{a_1} \left(\frac{u_1}{\rho_n} \right) 1_{(\frac{u_1-1}{\rho_n}, \frac{u_1}{\rho_n}]}(x).$$

Observe that $\lim_{n \rightarrow \infty} S_n(x) = g_{a_1}(x)$ and $|S_n(x)|$ is dominated by $g_{a_1}(\cdot)$. By applying DCT, we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} S_n(x) dx = \lim_{n \rightarrow \infty} \sum_{u_1=-(n-1)}^{n-1} g_{a_1}\left(\frac{u_1}{\rho_n}\right) \frac{1}{\rho_n} = \int_{-\infty}^{\infty} g_{a_1}(x) dx.$$

Thus, the upper bound of $nb_n T_3(\lambda_1, \lambda_2)$ given by (A.4) is $O(\rho_n b_n)$. Assumption ?? ensures that $nb_n T_3(\lambda_1, \lambda_2)$ converges to zero uniformly.

By combining all these terms, we have the convergence of $nb_n \text{Cov} \left[\text{Re}(\hat{\phi}_{a_1 a_2}(\lambda_1)), \text{Re}(\hat{\phi}_{a_3 a_4}(\lambda_2)) \right]$ as given in the theorem. Convergence of the other three covariances follow from a similar argument. \square

LEMMA 1. For $\lambda_1 - \lambda_2 = 0$, the function $T_{11}(\lambda_1, \lambda_2)$ converges as follows.

$$\lim_{n \rightarrow \infty} nb_n T_{11}(\lambda_1, \lambda_2) = \frac{1}{4} \left(\int_{-\infty}^{\infty} K^2(x) dx \right) \phi_{a_1 a_3}(\lambda_2) \phi_{a_2 a_4}^*(\lambda_2).$$

The convergence is uniform on any compact subset of the set

$$E = \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 = 0, -\infty < \lambda_1, \lambda_2 < \infty\}.$$

PROOF OF LEMMA 1. Consider a compact subset E' of the set E . Consider the simple function $S_n(\cdot)$, defined over $E' \times \mathbb{R}^3$ by

$$\begin{aligned} & S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) \\ &= \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(u_1 - u_2 + u_3)) \\ & \quad \times C_{a_1 a_3} \left(\frac{u_2}{\rho_n} \right) e^{-i \frac{u_2 \lambda_2}{\rho_n}} C_{a_2 a_4} \left(\frac{u_3}{\rho_n} \right) e^{i \frac{u_3 \lambda_2}{\rho_n}} 1_{((u_1-1)b_n, u_1 b_n]}(x_1) 1_{\left(\frac{(u_2-1)}{\rho_n}, \frac{u_2}{\rho_n}\right]}(x_2) 1_{\left(\frac{(u_3-1)}{\rho_n}, \frac{u_3}{\rho_n}\right]}(x_3). \end{aligned}$$

So that

$$nb_n T_{11}(\lambda_1, \lambda_2) = \frac{1}{(4\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3.$$

Define $u_{1n}(x_1)$, $u_{2n}(x_2)$ and $u_{3n}(x_3)$ as the smallest integers greater than or equal to x_1/b_n , $\rho_n x_2$ and $\rho_n x_3$, respectively. Thus, $(x_1, x_2, x_3) \in (b_n u_{1n-1}(x_1), b_n u_{1n}(x_1)] \times \left(\frac{u_{2n-1}(x_2)}{\rho_n}, \frac{u_{2n}(x_2)}{\rho_n}\right] \times \left(\frac{u_{3n-1}(x_3)}{\rho_n}, \frac{u_{3n}(x_3)}{\rho_n}\right]$ and $b_n u_{1n}(x_1) \rightarrow x_1$, $\frac{u_{2n}(x_2)}{\rho_n} \rightarrow x_2$, $\frac{u_{3n}(x_3)}{\rho_n} \rightarrow x_3$ as $n \rightarrow \infty$. Since $nb_n \rightarrow \infty$ and $b_n \rho_n \rightarrow 0$ as $n \rightarrow \infty$, we have, for any point $(x_1, x_2, x_3) \in \mathbb{R}^3$ and large enough n , the inequalities $-\frac{nb_n - x_1}{b_n \rho_n} < x_3 < \frac{nb_n - x_1}{b_n \rho_n}$, i.e., $-n + 1 - u_{1n}(x_1) < u_{3n}(x_3) < n - 1 - u_{1n}(x_1)$. Thus,

for sufficiently large n , we have

$$\begin{aligned} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) &= U_n(u_{1n}(x_1), u_{2n}(x_2), u_{3n}(x_3)) K(b_n u_{1n}(x_1)) K(b_n(u_{1n}(x_1) - u_{2n}(x_2) + u_{3n}(x_3))) \quad (\text{A.5}) \\ &\times C_{a_1 a_3} \left(\frac{u_{2n}(x_2)}{\rho_n} \right) e^{-i \frac{u_{2n}(x_2) \lambda_2}{\rho_n}} C_{a_2 a_4} \left(\frac{u_{3n}(x_3)}{\rho_n} \right) e^{i \frac{u_{3n}(x_3) \lambda_2}{\rho_n}}. \end{aligned}$$

Observe that, under Assumptions 1,3 and 4, the function $S_n(\lambda_1, \lambda_2, x_1, x_2, x_3)$ converges to the function $S(\cdot)$, defined over $E' \times \mathbb{R}^3$ by

$$S(\lambda_1, \lambda_2, x_1, x_2, x_3) = K^2(x_1) C_{a_1 a_3}(x_2) e^{-i x_2 \lambda_2} C_{a_2 a_4}(x_3) e^{i x_3 \lambda_2}.$$

Observe also that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3$ is a continuous function in (λ_1, λ_2) .

As in the proof of Theorem 1, we prove the convergence of the left hand side of (A.5) uniformly on E' , by showing that for any sequence $(\lambda_{1n}, \lambda_{2n}) \rightarrow (\lambda_1, \lambda_2)$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) dx_1 dx_2 dx_3 \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3. \end{aligned}$$

for $(\lambda_{1n}, \lambda_{2n}), (\lambda_1, \lambda_2) \in E'$. The latter convergence follows, through Assumption 1 and 2 and the DCT, from the dominance

$$|S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| \leq M K_1(x_1) g_{a_1 a_3}(x_2) g_{a_2 a_4}(x_3).$$

and the convergence of the integrand, which holds because of the continuity of $C_{a_1 a_3}(\cdot)$, $C_{a_2 a_4}(\cdot)$ and the kernel and the exponential functions. Hence, $nb_n T_{11}(\cdot)$ converges as stated uniformly on the compact set E' . \square

LEMMA 2. For $\lambda_1 - \lambda_2 \neq 0$, the function $nb_n T_{11}(\lambda_1, \lambda_2)$ converges to zero. The convergence is uniform on any compact subset of the set E_1 given by

$$E = \{(\lambda_1, \lambda_2) : \lambda_1 - \lambda_2 \neq 0, -\infty < \lambda_1, \lambda_2 < \infty\}.$$

PROOF OF LEMMA 2. Let E' be any compact subset of the set E . Consider the simple function $S_n(\cdot)$, defined over $E' \times \mathbb{R}^3$ by

$$\begin{aligned} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) &= \sum_{u_1=-(n-1)}^{(n-1)} \sum_{u_2=-(n-1)}^{(n-1)} \sum_{u_3=-(n-1)-u_1}^{(n-1)-u_1} U_n(u_1, u_2, u_3) K(b_n u_1) K(b_n(u_1 - u_2 + u_3)) e^{-i \frac{u_1(\lambda_1 - \lambda_2)}{\rho_n}} \\ &\times C_{a_1 a_3} \left(\frac{u_2}{\rho_n} \right) e^{-i \frac{u_2 \lambda_2}{\rho_n}} C_{a_2 a_4} \left(\frac{u_3}{\rho_n} \right) e^{i \frac{u_3 \lambda_2}{\rho_n}} 1_{((u_1-1)b_n, u_1 b_n]}(x_1) 1_{\left(\frac{(u_2-1)}{\rho_n}, \frac{u_2}{\rho_n}\right]}(x_2) 1_{\left(\frac{(u_3-1)}{\rho_n}, \frac{u_3}{\rho_n}\right]}(x_3). \end{aligned}$$

So that

$$nb_n T_{11}(\lambda_1, \lambda_2) = \frac{1}{(4\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3.$$

An argument similar to that used in the proof of Lemma 1 shows that for $(x_1, x_2, x_3) \in \mathbb{R}^3$ and sufficiently large n ,

$$\begin{aligned} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) &= U_n(u_{1n}(x_1), u_{2n}(x_2), u_{3n}(x_3)) K(b_n u_{1n}(x_1)) K(b_n(u_{1n}(x_1) - u_{2n}(x_2) + u_{3n}(x_3))) \\ &\times e^{-i \frac{u_{1n}(x_1)(\lambda_1 - \lambda_2)}{\rho_n}} C_{a_1 a_3} \left(\frac{u_{2n}(x_2)}{\rho_n} \right) e^{-i \frac{u_{2n}(x_2)\lambda_2}{\rho_n}} C_{a_2 a_4} \left(\frac{u_{3n}(x_3)}{\rho_n} \right) e^{i \frac{u_{3n}(x_3)\lambda_2}{\rho_n}}. \end{aligned}$$

where $u_{1n}(x_1)$, $u_{2n}(x_2)$ and $u_{3n}(x_3)$ are the smallest integers greater than or equal to x_1/b_n , $\rho_n x_2$ and $\rho_n x_3$, respectively.

For obtaining the uniform convergence of $nb_n T_{11}(\lambda_1, \lambda_2)$, consider

$$\begin{aligned} &\sup_{(\lambda_1, \lambda_2) \in E'} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3 \right| \\ &\leq \sup_{(\lambda_1, \lambda_2) \in E'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_n(\lambda_1, \lambda_2, x_1, x_2, x_3) - g_n(\lambda_1, \lambda_2, x_1, x_2, x_3)| dx_1 dx_2 dx_3 \\ &+ \sup_{(\lambda_1, \lambda_2) \in E'} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_n(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3 \right|, \end{aligned} \quad (\text{A.6})$$

where the function $g_n(\cdot)$ is defined over $E' \times \mathbb{R}^3$ by

$$g_n(\lambda_1, \lambda_2, x_1, x_2, x_3) = K^2(x_1) e^{-i \frac{x_1(\lambda_1 - \lambda_2)}{b_n \rho_n}} C_{a_1 a_3}(x_2) e^{-i x_2 \lambda_2} C_{a_2 a_4}(x_3) e^{i x_3 \lambda_2}.$$

We will show the uniform convergence of the right hand side of (A.6) by considering the two terms separately. For the first term, we follow the route taken in the proof of Theorem 1, i.e., show that for any sequence $(\lambda_{1n}, \lambda_{2n}) \rightarrow (\lambda_1, \lambda_2)$,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| dx_1 dx_2 dx_3 = 0$$

for $(\lambda_{1n}, \lambda_{2n}), (\lambda_1, \lambda_2) \in E'$. For this purpose, we write the above integral as

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| dx_1 dx_2 dx_3 \\ &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| dx_1 dx_2 dx_3 \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| dx_1 dx_2 dx_3, \end{aligned} \quad (\text{A.7})$$

where the function $G_n(\cdot)$ is defined over $E' \times \mathbb{R}^3$ by

$$G_n(\lambda_1, \lambda_2, x_1, x_2, x_3) = K^2(x_1) e^{-i \frac{u_{1n}(x_1) b_n (\lambda_1 - \lambda_2)}{b_n \rho_n}} C_{a_1 a_3}(x_2) e^{-i x_2 \lambda_2} C_{a_2 a_4}(x_3) e^{i x_3 \lambda_2}.$$

Now observe that

$$\begin{aligned} & |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| \\ & \leq M \left| e^{-i \frac{u_{1n}(x_1)b_n(\lambda_{1n}-\lambda_{2n})}{b_n\rho_n}} \alpha_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) \right|, \end{aligned}$$

where

$$\begin{aligned} & \alpha_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) \\ & = U_n(u_{1n}(x_1), u_{2n}(x_2), u_{3n}(x_3)) K(b_n u_{1n}(x_1)) K(b_n(u_{1n}(x_1) - u_{2n}(x_2) + u_{3n}(x_3))) \\ & \quad \times C_{a_1 a_3} \left(\frac{u_{2n}(x_2)}{\rho_n} \right) e^{-i \frac{u_{2n}(x_2)\lambda_{2n}}{\rho_n}} C_{a_2 a_4} \left(\frac{u_{3n}(x_3)}{\rho_n} \right) e^{i \frac{u_{3n}(x_3)\lambda_{2n}}{\rho_n}} \\ & \quad - K^2(x_1) C_{a_1 a_3}(x_2) e^{-ix_2 \lambda_{2n}} C_{a_2 a_4}(x_3) e^{ix_3 \lambda_{2n}}. \end{aligned}$$

Since $\alpha_n(\lambda_n, x, t, t') \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| = 0$$

Since from Assumption 1 and 2, we have the dominance

$$|S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| \leq 2MK_1(x_1)g_{a_1 a_3}(x_2)g_{a_2 a_4}(x_2).$$

By applying the DCT, we have

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| dx_1 dx_2 dx_3 = 0.$$

Turning to the second term on the right hand side of (A.7), observe that for any fixed x_1 ,

$$\left| e^{-i \frac{u_{1n}(x_1)b_n(\lambda_{1n}-\lambda_{2n})}{b_n\rho_n}} - e^{-i \frac{x_1(\lambda_{1n}-\lambda_{2n})}{b_n\rho_n}} \right| \leq \frac{\lambda_{1n} - \lambda_{2n}}{\rho_n}.$$

Thus,

$$\begin{aligned} & |G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| \\ & \leq M^2 g_{a_1 a_3}(0) g_{a_2 a_4}(0) \left| e^{-i \frac{u_{1n}(x_1)b_n(\lambda_{1n}-\lambda_{2n})}{b_n\rho_n}} - e^{-i \frac{x_1(\lambda_{1n}-\lambda_{2n})}{b_n\rho_n}} \right| \leq M^2 g_{a_1 a_3}(0) g_{a_2 a_4}(0) \frac{\lambda_{1n} - \lambda_{2n}}{\rho_n}, \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} |G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| = 0.$$

From Assumption 1 and 2, we have the dominance

$$|G_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3) - g_n(\lambda_{1n}, \lambda_{2n}, x_1, x_2, x_3)| \leq 2MK_1(x_1)g_{a_1 a_3}(x_2)g_{a_2 a_4}(x_2).$$

which leads us, through another use of the DCT, to the convergence of the second integral of (A.7). This establishes that the first term on the right hand side of (A.6) converges to 0. We only have to deal with the second term. Let

$$s_n(\lambda_1, \lambda_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_n(\lambda_1, \lambda_2, x_1, x_2, x_3) dx_1 dx_2 dx_3.$$

In order to establish the uniform convergence of $s_n(\cdot)$ over E' , it is enough to show that $s_n(\lambda_{1n}, \lambda_{2n}) \rightarrow 0$ for any sequence $(\lambda_{1n}, \lambda_{2n}) \rightarrow (\lambda_1, \lambda_2)$, where $(\lambda_{1n}, \lambda_{2n}), (\lambda_1, \lambda_2) \in E'$. By using the Reimann-Lebesgue lemma, we have $s_n(\lambda_1, \lambda_2) \rightarrow 0$. Thus, the second term on the right hand side of (A.6) also converges to 0. Hence, $nb_n T_{11}(\lambda_1, \lambda_2)$ converges to 0 uniformly on E' as $n \rightarrow \infty$. \square

PROOF OF THEOREM 3. $cum(\hat{\phi}_{a_1 a_2}(\lambda_1), \hat{\phi}_{a_3 a_4}(\lambda_2), \dots, \hat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L))$ can be written as

$$\begin{aligned} & cum(\hat{\phi}_{a_1 a_2}(\lambda_1), \hat{\phi}_{a_3 a_4}(\lambda_2), \dots, \hat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L)) \\ &= \frac{1}{(\pi n \rho_n)^L} \sum_{t_1=1}^n \sum_{t_2=1}^n \dots \sum_{t_{2L-1}=1}^n \sum_{t_{2L}=1}^n K(b_n(t_1 - t_2)) \dots K(b_n(t_{2L-1} - t_{2L})) e^{-\frac{i(t_1 - t_2)\lambda_1}{\rho_n}} \times \dots \\ & \times e^{-\frac{i(t_{2L-1} - t_{2L})\lambda_L}{\rho_n}} cum\left(X_{a_1}\left(\frac{t_1}{\rho_n}\right) X_{a_2}\left(\frac{t_2}{\rho_n}\right), \dots, X_{a_{2L-1}}\left(\frac{t_{2L-1}}{\rho_n}\right) X_{a_{2L}}\left(\frac{t_{2L}}{\rho_n}\right)\right) \end{aligned} \quad (\text{A.8})$$

It follows that

$$\begin{aligned} & |cum(\hat{\phi}_{a_1 a_2}(\lambda_1), \hat{\phi}_{a_3 a_4}(\lambda_2), \dots, \hat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L))| \\ & \leq \frac{1}{(n \rho_n)^L} \sum_{t_1=1}^n \sum_{t_2=1}^n \dots \sum_{t_{2L-1}=1}^n \sum_{t_{2L}=1}^n |K(b_n(t_1 - t_2)) \dots K(b_n(t_{2L-1} - t_{2L}))| \\ & \quad \times \left| cum\left(X_{a_1}\left(\frac{t_1}{\rho_n}\right) X_{a_2}\left(\frac{t_2}{\rho_n}\right), \dots, X_{a_{2L-1}}\left(\frac{t_{2L-1}}{\rho_n}\right) X_{a_{2L}}\left(\frac{t_{2L}}{\rho_n}\right)\right) \right| \end{aligned}$$

Now

$$\begin{aligned} & cum\left(X_{a_1}\left(\frac{t_1}{\rho_n}\right) X_{a_2}\left(\frac{t_2}{\rho_n}\right), \dots, X_{a_{2L-1}}\left(\frac{t_{2L-1}}{\rho_n}\right) X_{a_{2L}}\left(\frac{t_{2L}}{\rho_n}\right)\right) \\ &= \sum_{\nu} C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1k_1}}} \left(\frac{t_{j_{11}} - t'_1}{\rho_n}, \dots, \frac{t_{j_{1,k_1-1}} - t'_1}{\rho_n} \right) \dots \\ & \quad \times C_{a_{j_{P1}} a_{j_{P2}} \dots a_{j_{Pk_P}}} \left(\frac{t_{j_{P1}} - t'_P}{\rho_n}, \dots, \frac{t_{j_{P,k_P-1}} - t'_P}{\rho_n} \right) \end{aligned}$$

where the summation is over all *indecomposable* (Brillinger, 2001; Leonov and Shirayev, 1959) partitions $\nu = (\nu_1, \dots, \nu_P)$, such that $\nu_p = (j_{p1}, \dots, j_{pk_p}), p = 1, \dots, P$, of the table

1	2
3	4
\vdots	\vdots
2L-1	2L

and $t'_p = t_{j_{pk_p}}, p = 1, \dots, P$. Since the partition ν is indecomposable, we have

$$t_{j_{pl}} - t'_p \neq t_{2m} - t_{2m-1}; \quad l = 1, \dots, k_p; \quad p = 1, \dots, P; \quad m = 1, \dots, L.$$

Define

$$u_{j_{pl}} = t_{j_{pl}} - t'_p; \quad l = 1, \dots, k_p; \quad p = 1, \dots, P.$$

Note that $u_{j_{pk_p}} = 0$ for $p = 1, \dots, P$. Then the joint cumulant of $(\widehat{\phi}_{a_1 a_2}(\lambda_1), \widehat{\phi}_{a_3 a_4}(\lambda_2), \dots, \widehat{\phi}_{a_{2L-1} a_{2L}}(\lambda_L))$ given by (A.8) is absolutely bounded by

$$\begin{aligned} & \frac{1}{(n\rho_n)^L} \sum_{\nu} \sum_{t'_1=1}^n \sum_{u_{j_{11}}=-(t'_1-1)}^{n-t'_1} \cdots \sum_{u_{j_{1,k_1-1}}=-(t'_1-1)}^{n-t'_1} \cdots \sum_{t'_P=1}^n \sum_{u_{j_{P1}}=-(t'_P-1)}^{n-t'_P} \cdots \sum_{u_{j_{P,k_P-1}}=-(t'_P-1)}^{n-t'_P} \\ & \left| K[b_n(u_1 + t'_{p_1} - u_2 - t'_{p_2})] \cdots K[b_n(u_{2L-1} + t'_{p_{2L-1}} - u_{2L} - t'_{p_{2L}})] \right| \\ & \times \left| C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1k_1}}} \left(\frac{u_{j_{11}}}{\rho_n}, \dots, \frac{u_{j_{1,k_1-1}}}{\rho_n} \right) \cdots C_{a_{j_{P1}} a_{j_{P2}} \dots a_{j_{Pk_P}}} \left(\frac{u_{j_{P1}}}{\rho_n}, \dots, \frac{u_{j_{P,k_P-1}}}{\rho_n} \right) \right|, \end{aligned} \quad (\text{A.9})$$

where p_m is that member of the set $\{1, 2, \dots, P\}$ which satisfies $t_m \in \nu_{p_m}$ for $m = 1, \dots, L$.

We will now show that the set $A = \{t'_{p_1} - t'_{p_2}, \dots, t'_{p_{2L-1}} - t'_{p_{2L}}\}$ has $P-1$ linearly independent elements. Note that the set A consists of differences of pairs of elements of the set $\{t'_1, t'_2, \dots, t'_P\}$. So the set A can have at most $P-1$ linearly independent differences. Suppose that the set A has exactly $P-j$ linearly independent differences for some $j \geq 1$. Denote the $P-j$ independent differences of the set A by

$$A_1 = \left\{ t'_{p_{2k_1-1}} - t'_{p_{2k_1}}, t'_{p_{2k_2-1}} - t'_{p_{2k_2}}, \dots, t'_{p_{2k_{P-j-1}-1}} - t'_{p_{2k_{P-j-1}}} \right\},$$

where $k_1, \dots, k_{P-j} \in \{1, 2, \dots, L\}$. Let, if possible, $j > 1$, and consider a difference $t'_{l_1} - t'_{l_2}$ for $l_1, l_2 \in \{1, 2, \dots, P\}$ which is linearly independent of the elements of the set A_1 . Since the partition ν is indecomposable, the sets ν_{l_1} and ν_{l_2} *communicate* (Leonov and Shirayayev, 1959). Therefore, there exists an index set $\{s_1, s_2, \dots, s_r\}$ with $r \geq 2$, which is a proper subset of $\{1, 2, \dots, P\}$, such that $s_1 = l_1$, $s_r = l_2$ and the pairs $(\nu_{s_1}, \nu_{s_2}), (\nu_{s_2}, \nu_{s_3}), \dots, (\nu_{s_{r-1}}, \nu_{s_r})$ are *hook* (Leonov and Shirayayev, 1959). Consequently, there exist indices $j_1, \dots, j_{r-1} \in \{1, \dots, L\}$ such that for $m = 1, \dots, r-1$, one of the points $t_{2j_{m-1}}$ and t_{2j_m} belongs to ν_{s_m} and the other belongs to $\nu_{s_{m+1}}$. It follows that for $m = 1, \dots, r-1$, $(t'_{p_{2j_{m-1}}} - t'_{p_{2j_m}})$ is in A , and hence, they can be written as linear combinations of the members of A_1 . Note that for $m = 1, \dots, r-1$, $(t'_{s_{m-1}} - t'_{s_m})$ is equal to either $(t'_{p_{2j_{m-1}}} - t'_{p_{2j_m}})$ or $-(t'_{p_{2j_{m-1}}} - t'_{p_{2j_m}})$. Thus,

$$t'_{l_1} - t'_{l_2} = t'_{s_1} - t'_{s_r} = (t'_{s_1} - t'_{s_2}) + (t'_{s_2} - t'_{s_3}) + \cdots + (t'_{s_{r-1}} - t'_{s_r})$$

can be written as a linear combination of the members of A_1 . This fact contradicts the assumption that $t'_{l_1} - t'_{l_2}$ is linearly independent of the elements of the set A_1 . Therefore, j cannot be larger than 1. This proves that the set A cannot contain fewer than $P - 1$ linearly independent differences.

Consider the $P - 1$ linearly independent elements of the set A_1 , where $j = 1$, and define

$$\begin{aligned} v_1 &= u_{2k_1-1} + t'_{p_{2k_1-1}} - u_{2k_1} - t'_{p_{2k_1}}, \\ &\vdots \\ v_{P-1} &= u_{2k_{P-1}-1} + t'_{p_{2k_{P-1}-1}} - u_{2k_{P-1}} - t'_{p_{2k_{P-1}}}. \end{aligned}$$

Using the above transformation, and by replacing the P sums over indices t'_1, \dots, t'_P by $P - 1$ sums over the indices v_1, \dots, v_{P-1} , we find that the joint cumulant given in (A.9) is bounded from above by

$$\begin{aligned} &\frac{1}{n^{L-1} \rho_n^L} \sum_{\mathbf{v}} M^{L-P+1} \sum_{u_{j11}=-(n-1)}^{n-1} \dots \sum_{u_{j1,k_1-1}=-(n-1)}^{n-1} \dots \sum_{u_{jP1}=-(n-1)}^{n-1} \dots \sum_{u_{jP,k_P-1}=-(n-1)}^{n-1} \\ &\sum_{v_1=-3n}^{3n} \dots \sum_{v_{P-1}=-3n}^{3n} |K(b_n v_1)| \dots |K(b_n v_{P-1})| \\ &\times \left| C_{a_{j11} a_{j12} \dots a_{j1k_1}} \left(\frac{u_{j11}}{\rho_n}, \dots, \frac{u_{j1,k_1-1}}{\rho_n} \right) \right| \dots \left| C_{a_{jP1} a_{jP2} \dots a_{jPk_P}} \left(\frac{u_{jP1}}{\rho_n}, \dots, \frac{u_{jP,k_P-1}}{\rho_n} \right) \right|. \end{aligned} \quad (\text{A.10})$$

The above simplification has been made by taking into account the upper bound for $L - P + 1$ copies of $K(\cdot)$ and conservative estimates of the ranges of summation of v_1, \dots, v_{P-1} . Now one can rewrite the expression in (A.10) as follows.

$$\begin{aligned} &\sum_{\mathbf{v}} M^{L-P+1} \frac{(\rho_n b_n)^{L-P}}{(n b_n)^{L-1}} \left[\sum_{v_1=-3n}^{3n} K(b_n v_1) b_n \right] \dots \left[\sum_{v_{P-1}=-3n}^{3n} K(b_n v_{P-1}) b_n \right] \\ &\times \left\{ \sum_{u_{j11}=-(n-1)}^{n-1} \dots \sum_{u_{j1,k_1-1}=-(n-1)}^{n-1} \left| C_{a_{j11} a_{j12} \dots a_{j1k_1}} \left(\frac{u_{j11}}{\rho_n}, \dots, \frac{u_{j1,k_1-1}}{\rho_n} \right) \right| \frac{1}{\rho_n^{k_1-1}} \right\} \dots \\ &\times \left\{ \sum_{u_{jP1}=-(n-1)}^{n-1} \dots \sum_{u_{jP,k_P-1}=-(n-1)}^{n-1} \left| C_{a_{jP1} a_{jP2} \dots a_{jPk_P}} \left(\frac{u_{jP1}}{\rho_n}, \dots, \frac{u_{jP,k_P-1}}{\rho_n} \right) \right| \frac{1}{\rho_n^{k_P-1}} \right\}. \end{aligned} \quad (\text{A.11})$$

Consider the simple function $S_n(\cdot)$ defined over \mathbb{R} by

$$S_n(x) = \sum_{v_1=-3n}^{3n} K(b_n v_1) 1_{(b_n v_1-1, b_n v_1]}(x).$$

Note that $\int_{-\infty}^{\infty} S_n(x)dx = \sum_{v_1=-3n}^{3n} K(b_n v_1) b_n$, and from Assumption 2 we have the dominance $S_n(x) \leq K_1(x)$. By applying the DCT, we have

$$\sum_{v_1=-3n}^{3n} K(b_n v_1) b_n \rightarrow \int_{-\infty}^{\infty} |K(x)| dx.$$

This fact establishes the convergence of the sums over v_1, \dots, v_{P-1} .

Consider the simple function $T_n(\cdot)$ defined over \mathbb{R}^{k_1-1} by

$$\begin{aligned} T_n(x_1, x_2, \dots, x_{k_1-1}) &= \sum_{u_{j_{11}}=-(n-1)}^{n-1} \dots \sum_{u_{j_{1,k_1-1}}=-(n-1)}^{n-1} C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1,k_1}}} \left(\frac{u_{j_{11}}}{\rho_n}, \dots, \frac{u_{j_{1,k_1-1}}}{\rho_n} \right) \\ &\quad \times 1\left(\frac{u_{j_{11}}-1}{\rho_n}, \frac{u_{j_{11}}}{\rho_n}\right] (x_1) \dots 1\left(\frac{u_{j_{1,k_1-1}}-1}{\rho_n}, \frac{u_{j_{1,k_1-1}}}{\rho_n}\right] (x_{k_1-1}). \end{aligned}$$

Note that

$$\begin{aligned} &\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} T_n(x_1, \dots, x_{k_1-1}) dx_1 \dots dx_{k_1-1} \\ &= \sum_{u_{j_{11}}=-(n-1)}^{n-1} \dots \sum_{u_{j_{1,k_1-1}}=-(n-1)}^{n-1} \left| C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1,k_1}}} \left(\frac{u_{j_{11}}}{\rho_n}, \dots, \frac{u_{j_{1,k_1-1}}}{\rho_n} \right) \right| \frac{1}{\rho_n^{k_1-1}}. \end{aligned}$$

From Assumption 5A, we have that the function $T_n(\cdot)$ is bounded by an integrable function. Thus, by applying the DCT, we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{u_{j_{11}}=-(n-1)}^{n-1} \dots \sum_{u_{j_{1,k_1-1}}=-(n-1)}^{n-1} \left| C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1,k_1}}} \left(\frac{u_{j_{11}}}{\rho_n}, \dots, \frac{u_{j_{1,k_1-1}}}{\rho_n} \right) \right| \frac{1}{\rho_n^{k_1-1}} \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left| C_{a_{j_{11}} a_{j_{12}} \dots a_{j_{1,k_1}}} (x_1, \dots, x_{k_1-1}) \right| dx_1 \dots dx_{k_1-1}. \end{aligned}$$

Likewise, we have the convergence for the remaining $P-1$ sets of summations. Using these above convergence results, the upper bound of (A.9) given in (A.11) can be written as

$$\sum_{\boldsymbol{\nu}} \frac{(\rho_n b_n)^{L-P}}{(n b_n)^{L-1}} d_{\boldsymbol{\nu}},$$

where $d_{\boldsymbol{\nu}}$ are appropriate constants. The summation is over the finite number of indecomposable partitions, and the worst-case value of the partition size P is L . Therefore, the upper bound is $O((n b_n)^{-(L-1)})$. This Completes the proof of Theorem 3. \square

PROOF OF THEOREM 4. Note that the first moment of the random vector on the left hand side of (5) is zero and the second moment converges in accordance with Theorem 2. Further,

$$\text{cum}(c_1(Y_1 - d_1), c_1(Y_2 - d_2), \dots, c_J(Y_J - d_J)) = c_1 c_2 \dots c_J \times \text{cum}(Y_1, Y_2, \dots, Y_J),$$

for any set of constants $c_1, \dots, c_J, d_1, \dots, d_J$. From the above fact and Theorem 3, for all $k > 2$, the absolute value of the k th order joint cumulant of the random vector on the left hand side of (5) is bounded from above by an $O((nb_n)^{k/2-k+1})$ term. According to Assumption 3, this upper bound tends to 0 as n tends to infinity. This completes the proof. \square

PROOF OF THEOREM 5. The result can be proved along the lines of the proof of Theorem 3 of Srivastava and Sengupta (2010). \square

PROOF OF THEOREM 6. The weak convergence of the first term on the right hand side of (6) follows from Theorem 4. On the other hand, the second term can be written, in view of Theorem 5, as

$$\sqrt{nb_n} \left(E[\widehat{\phi}_{a_1 a_2}(\lambda_1)] - \phi_{a_1 a_2}(\lambda_1) \right) = \sqrt{nb_n} \left(O((\rho_n b_n)^q) + O\left(\frac{\rho_n}{n}\right) + O\left(\frac{1}{\rho_n^p}\right) \right). \quad (\text{A.12})$$

Under Assumption 3,

$$\lim_{n \rightarrow \infty} \sqrt{nb_n} \rho_n^q b_n^q = 0 \Rightarrow \lim_{n \rightarrow \infty} \sqrt{nb_n} \frac{\rho_n}{n} = 0.$$

Therefore, under Assumptions 3 and 4A, the right hand side of (A.12) goes to zero as $n \rightarrow \infty$. This completes the proof. \square

PROOF OF THEOREM 7. Note that under Assumption 4A, we have

$$\lim_{n \rightarrow \infty} \sqrt{nb_n} \frac{1}{\rho_n^p} = 0 \quad (\text{A.13})$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{nb_n} \rho_n^q b_n^q = 0 & \Leftrightarrow \lim_{n \rightarrow \infty} (nb_n)^{\frac{1}{2q}} b_n \rho_n = 0 \\ \Leftrightarrow \lim_{n \rightarrow \infty} (nb_n)^{1+\frac{1}{2q}} \frac{\rho_n}{n} = 0 & \Leftrightarrow \lim_{n \rightarrow \infty} \sqrt{nb_n} \left(\frac{\rho_n}{n} \right)^{\frac{q}{1+2q}} = 0. \end{aligned} \quad (\text{A.14})$$

From (A.13) and (A.14), we have

$$\frac{1}{\sqrt{nb_n}} = o\left(\left(\frac{\rho_n}{n}\right)^{\frac{q}{1+2q}}\right), \quad (\text{A.15})$$

$$\text{and } \frac{1}{\sqrt{nb_n}} = o\left(\left(\frac{1}{\rho_n}\right)^p\right). \quad (\text{A.16})$$

The right hand sides of (A.15) and (A.16) are increasing and decreasing functions, respectively, of ρ_n . Assumption 3, together with (A.13), indicate that ρ_n goes to infinity as n goes to infinity. The rate given by (A.15) will be unduly slow if ρ_n goes to infinity too slowly, while the rate given by (A.16) will be unduly slow if ρ_n goes to infinity too fast. At either event, $1/\sqrt{nb_n}$ will have a sub-optimal rate of convergence to zero. It follows that $1/\sqrt{nb_n}$ has the fastest convergence to zero if

$$O\left(\left(\frac{n}{\rho_n}\right)^{\frac{q}{1+2q}}\right) = O(\rho_n^p).$$

This condition requires that $\rho_n = O\left(n^{-\frac{q}{p+q+2pq}}\right)$. For this rate of ρ_n , (A.15) implies that

$$b_n = o\left(n^{-\frac{p+q}{p+q+2pq}}\right) \quad \text{and} \quad \frac{1}{\sqrt{nb_n}} = o\left(n^{-\frac{pq}{p+q+2pq}}\right).$$

This completes the proof. □

References

- BRILLINGER, D. R. (2001). *Time Series Data Analysis and Theory*. Philadelphia: SIAM.
- BROCKWELL, P. J. and DAVIS, R. A. (1991). *Time Series: Theory and Methods*. New York: Springer-Verlag.
- CHEN, H., SIMPSON, D. G. and YING, Z. (2000). Infill asymptotics for a stochastic process model with measurement error. *Statist. Sinica* 10, 141-156.
- CONSTANTINE, A. G. and HALL, P. (1994). Characterizing surface smoothness via estimation of effective fractal dimension. *J. Roy. Statist. Soc. Ser. B* 56, 96-113.
- FUENTES, M. (2002). Spectral methods for nonstationary spatial processes. *Biometrika* 89, 197-210.
- HALL, P., FISHER, N. I. and HOFFMANN, B. (1994). On the nonparametric estimation of covariance functions. *Ann. Statist.* 22, 2115-2134.
- HOEL, P. G., PORT, S. C. and STONE, C. J. (1972). *Introduction to Stochastic Processes*. Boston: Houghton Mifflin.
- KAY, S. M. (1999). *Modern Spectral Estimation: Theory and Application*. Englewood Cliffs, New Jersey: Prentice Hall.
- LAHIRI, S. N. (1999). Asymptotic distribution of the empirical spatial cumulative distribution function predictor and prediction bands based on a subsampling method. *Probab. Theory Related Fields* 114, 55-84.
- LEONOV, V. P. and SHIRYAYEV, A. N. (1959). On a method of calculation of semi-invariants. *Theory Probab. Appl.* 4, 319-329.
- LIM, C. Y. and STEIN, M. (2008). Properties of spatial cross-periodograms using fixed-domain asymptotics. *J. Multivariate Anal.* 99, 1962-1984.

- MARVASTI, F. A. (2001). *Nonuniform Sampling*. New York: Kluwer Plenum.
- MASRY, E. (1978). Alias-free sampling: An alternative conceptualization and its application. *IEEE Trans. Inf. Theor.* IT-24, 173-183.
- PARZEN, E. (1957). On consistent estimation of the spectrum of the stationary time series. *Annals of Mathematical Statistics* 28, 329-348.
- RESNICK, S. I. (1987). *Extreme Values, Regular Variation and Point Processes*. New York : Springer-Verlag.
- SHAPIRO, H. S. and SILVERMAN, R. A. (1960). Alias-free sampling of random noise. *J. Soc. Indust. Appl. Math.* 8, 225-248.
- SHORAK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics*. New York: John Wiley.
- SRIVASTAVA, R. and SENGUPTA, D. (2010). Consistent estimation of non-bandlimited spectral density from uniformly spaced samples. to appear in *IEEE Trans. Inf. Theor.*, (preprint available in `\protect\vrule width0pt\protect\href{http://arxiv.org/abs/0906.5045}`{arXiv:0906.5045}).
- STEIN, M. L. (1995). Fixed-domain asymptotics for spatial periodograms. *J. Amer. Statist. Assoc.* 90, 1277-1288.
- ZHANG, H. and ZIMMERMAN, D. L. (2005). Towards reconciling two asymptotic frameworks in spatial statistics. *Biometrika* 92, 921-936.